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THE VON MISES PLASTICALLY DEFORMED ENCLAVE AS HEAT SOURCE FOR RUNNING CRACKS

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Abstract—All existing models interrelating crack propagation characteristic properties with heat exchange in zones around the crack tips simulate the heat production area by simple geometrical forms, like circles or squares. They assume also constant heat production rates over the whole area of the source. Contrariwise, the model introduced in the present study is based on two improved and more realistic assumptions concerning the shape and dimensions of the heat source and the spatial distribution of heat production density inside the heat source. These assumptions, together with Rosenthal's moving-point-source solution, yield a reasonable and improved model for a fast and rather simple numerical approach, whose results are in agreement with existing experimental evidence. The method was applied to two different materials, one polymer (polycarbonate) and the other metal (aluminum alloy) and their results appear to be compatible with reality and concordant with respective experiments. In addition, the method was applied to relatively high crack propagation velocities revealing the existence of two symmetric off-axis temperature extrema in either side of the crack propagation axis. This behavior may be directly related to the phenomenon of crack branching, where similar maxima of energies and directions of branching are systematically observed at the same velocities.

1. INTRODUCTION

It is widely accepted that heat exchange is always present during crack propagation. The source of this heat is the area around the travelling crack tip, which is in general plastically deformed. Indeed, plastic deformations are the cause of the production of high heat concentrations around the crack tips. It is well known that a fraction of about 60–80% of the total plastic energy stored in polymeric materials is converted into heat (Engleter and Müller, 1958), while this fraction reaches 90% for metals (Taylor and Quinney, 1958).

Taking into account the fact that, even for brittle materials, the energy consumed for the creation of new surfaces is negligible as compared with the respective plastic energy, the same being true for the kinetic energy of the small plastic zone, we can affirm that a portion of about 60–90% of the total energy, furnished to the plastically deformed zone by the elastic stress field, is released in the form of heat. This heat concentration causes significant temperature increases in the immediate vicinity of the running crack tip. The great interest for the determination of the temperature field surrounding the plastic zone is justified, if we consider that a rise of the temperature only of the order of some decades reduces drastically the value of the mechanical properties of any material. For example, an increase of the temperature from about 20°C to about 80°C causes a decrease of the yield stress of PMMA to half its initial value (Bowden, 1973), whereas a temperature increase of 400°C in steels creates a 30% decrease of the value of the elastic modulus (Theocaris and Coroneos, 1964).

However, despite the effort invested until now, the exact temperature field is still unknown, and large dispersions between the reported values have been recorded. Fox and Fuller (1971), studying the infra-red radiation emitted during the fracture of PMMA specimens, have recorded temperature elevations of the order of 500°C, whereas Weichert and Schönert (1978), estimated temperatures between 2500°K and 3000°K for glass and about 4000°K for quartz by analyzing the light emission observed during fracture and comparing the resulting relative light intensities with the normalized black body radiations. Moreover, the theoretical model of Weichert and Schönert (1974), predicted maximum temperature rises of 910°K for crack velocities of 200 m s^{-1} and for radii of the circular heat zones of the order of $\rho = 20\text{Å}$, while at the front of the heated zones the predicted temperature elevations were lower than 300 K. Contrariwise, the approach of Marshall *et al.* (1974) gives for PMMA specimens much lower temperature elevations.

However, it is worthwhile emphasizing that all these values are merely indicative, since the exact temperatures are functions of the coordinates of the points examined, relative to the crack tips and, also, of the dimensions and the velocities of the heat sources. Moreover the discrepancies observed are partly due to the various rather inadequate models simulating the heat sources. Indeed, the shapes and dimensions of the heat sources and the forms of the heat-production distribution affect strongly the final results, as it is noted in Weichert and Schönert (1974). The existing models simulate the heat-production areas by simple geometrical forms, like circles or squares, of rather arbitrary dimensions, assuming also constant heat production rates over the whole area of the source. These assumptions constitute the weaknesses of the existing models. The model introduced in this paper is relieved from these assumptions and therefore constitutes an improved description of the phenomenon of heat production around the tip of a running crack in physically sound manner.

2. DESCRIPTION OF THE MODEL

The model is based on two basic hypotheses, simulating better the heat-production phenomenon.

Hypothesis 2.1. *The shape of the heat source coincides with the shape of the plastically deformed zone enveloping the moving crack tip.* Since heat is mainly produced inside plastic zones, this first assumption approaches reality. The bounds of the plastic enclaves are determined by means of a proper yield condition.

Hypothesis 2.2. *The density at each point of the heat source is intimately related to the amount of plastic deformation at the same point.* This assumption implies that the heat density depends on the local value of the respective equivalent stress.

Thus, the heat source is readily determined by the well-known von Mises yield criterion for plane stress, which is intimately related with the component of the distortional energy density. The von Mises criterion is expressed by

$$\sigma_{xx}^2(r,\theta)+\sigma_{yy}^2(r,\theta)+3\sigma_{xy}^2(r,\theta)-\sigma_{xx}(r,\theta)\sigma_{yy}(r,\theta) = \sigma_0^2 \tag{1}$$

where $\sigma_{ij}(r,\theta)$, $(i, j = x, y)$ are the components of the dynamic stress field, as it is described by Freund and Clifton (1974). Moreover, (r,θ) are the polar coordinates centered on the moving crack-tip and σ_0 is the yield stress in simple tension of the material. Solving eqn (1) with respect to r, the following expression for the radius, $r_m(\theta)$, of the elastic–plastic boundary is obtained:

$$r_m(\theta) = \{a_1 f_1^2(\theta)+a_2 f_2^2(\theta)+a_3 f_1(\theta)f_2(\theta)+a_4[f_3(\theta)-f_4(\theta)]^2\} \tag{2}$$

where the functions $f_i(\theta)$, $(i = 1, 4)$ depend upon the components of the dilatational (c_1) and the distortional (c_2) wave velocities of the material, and the constants a_i $(i = 1, 4)$, depend upon the velocity of the crack (c).

The $f_i(\theta)$ functions are expressed by

$$f_{1,3} = \left[\frac{1}{(1-c^2/c_1^2\sin^2\theta)^{1/2}} \pm \frac{\cos\theta}{1-c^2/c_1^2\sin^2\theta}\right]^{1/2}$$

$$f_{2,4} = \left[\frac{1}{(1-c^2/c_2^2\sin^2\theta)^{1/2}} \pm \frac{\cos\theta}{1-c^2/c_2^2\sin^2\theta}\right]^{1/2}. \tag{3}$$

Moreover, the functions a_i $(i = 1, 4)$ are given by

$$a_1 = \mathscr{G}\left[(1+2S_1^2-S_2^2)^2+\left(\frac{4S_1S_2}{1+S_2^2}\right)^2-(1+2S_1^2-S_2^2)^2(1+S_2^2)\right]$$

$$a_2 = \mathscr{G}\left(\frac{4S_1S_2}{1+S_2^2}\right)^2, \qquad a_3 = -3\mathscr{G}\frac{8S_1S_2}{1+S_2^2}(S_1^2-S_2^2), \qquad a_4 = 12\mathscr{G}S_1^2 \tag{4}$$

with

$$S_1^2 = 1-(c/c_1)^2, \qquad S_2^2 = 1-(c/c_2)^2$$

and

$$\mathscr{G} = \frac{1+S_2^2}{[4S_1S_2-(1+S_2^2)^2]}\cdot\frac{K_1^2}{\pi\delta_0^2}. \tag{5}$$

Relation (5) expressing the mode I energy rate $\mathscr{G}$ for dynamic loading, whose static term is defined in McClintock, and Irwin (1965), takes into account the effect of yielding upon the stress distribution near the crack tip. K_1 is the mode I dynamic stress intensity factor which depends on its static value K_{1S} multiplied by a correction factor defined in Rose (1976).

This distribution, as derived from a mode III elastic–plastic analysis, indicates that yielding for elastic–perfectly plastic materials translates the stress curve away from the crack tip by an amount equal to the radius of the plastic zone, as it is derived from the elastic analysis. Thus, the plastic zone is assumed to be extended radially by a factor of two. For mode I loadings and for brittle or semi-brittle materials under dynamic loading, this factor of two is reduced considerably. However, here it is taken equal to 2 so that the energy rate $\mathscr{G}$ is doubled in relation (5).

The spatial distribution of heat-production density is $dq(r_s, \theta_s)$, where r_s, θ_s are the coordinates of a generic point S inside the heat source, and it is defined by a convenient function covering the following requirements:

(i) The function must give a maximum heat density at the crack tip ($r_s = 0$) and zero heat density along the elastic–plastic boundary [$r_s = r_m(\theta)$].
(ii) Between the above two extreme values the function must follow a law similar to the variation of the equivalent stress inside the plastic enclave, which is analogous to plastic work.
(iii) The integral of $dq(r_s, \theta_s)$ over the source surface A must give the total heat production Q, of the source.

A convenient family of functions satisfying the two above cited conditions is expressed by

$$dq(r_s, \theta_s) = \frac{\kappa}{A}\left[\frac{2r_m^n}{r_s^n+r_m^n}-1\right]\cos(\theta/2)\,dA \tag{6}$$

where A is the total heat source area, dA is the area of each elementary heat source inside the plastic zone, and n is a positive arbitrary constant. Moreover, κ is a parameter evaluated by satisfying the third condition. The total heat production over the whole area, A, of the source must be constant and independent of the type of grid network of sources and the number of their partitions inside the plastic enclave, selected for the numerical evaluation of the temperature rise at any point outside the plastic region. Then it is valid that:

$$\iint dq(r_s, \theta_s) = Q \tag{7}$$

or, by means of eqn (3) we may deduce that:

$$\kappa = \frac{QA}{2}\left\{\int_0^{\pi}\int_0^{r_m}\left(\frac{2r_m^n}{r_s^n+r_m^n}-1\right)r_s\cos(\theta/2)\,dr_s\,d\theta\right\}^{-1} \tag{8}$$

where r_m stands for $r_m(\theta)$. It was taken into account in this relation that the axis of the crack is the axis of symmetry of the source for an external load perpendicular to the crack axis.

For simplicity, the exponent n was taken as $n = 1$, since the equivalent stress distribution inside the plastic enclave may be accepted as obeying a r^{-1} law. However, the influence of this exponent shall be studied in the sequel. The double integration of the denominator of eqn (8), with $n = 1$, can be executed numerically, but since the analytical expression of the elastic–plastic radius $r_m(\theta)$ is known from eqn (2), we can, after some algebra, convert the double integrals into single integrals, thus avoiding the various error sources interfering during the numerical evaluation of the surface integrals. The final form of the factor κ is given by

$$\kappa = \frac{QA}{\left[(3-4\ln 2)\int_0^{\pi} r_m^2(\theta)\cos(\theta/2)\,d\theta\right]}. \tag{9}$$

Introducing eqns (2) and (9) into eqn (6) we obtain for the heat-density distribution $dq(r_s, \theta_s)$ the expression:

$$dq(r_s,\theta_s) = \frac{Q\left(\frac{2r_m}{r_s+r_m}-1\right)\cos\frac{\theta}{2}\,dA}{(3-4\ln 2)\int_0^{\pi}\{a_1 f_1^2(\theta)+a_2 f_2^2(\theta)+a_3 f_1(\theta)f_2(\theta)+a_4[f_3(\theta)-f_4(\theta)]^2\}^{-2}\cos(\theta/2)\,d\theta} \tag{10}$$

whose numerical evaluation is elementary.

In order to calculate, now, the temperature rise at a point $P(r_p, \theta_p)$, we divide the heat source into a number of elementary heat sources $S(r_s, \theta_s)$, where (r_p, θ_p) and (r_s, θ_s) are the polar coordinates of the point examined and of the elementary heat source with respect to the moving crack tip, respectively, as it is indicated in Fig. 1. Each elementary heat source is assumed to be a moving-point heat source, which causes at the fixed generic point $P(r_p, \theta_p)$ an elementary temperature rise $d(\Delta T)$ (r_p, θ_p), given by the well-known Rosenthal's (1946) equation:

$$d(\Delta T)(r_p,\theta_p) = \frac{c\,dq(r_s,\theta_s)}{2\pi\mathrm{K}}\left\{\exp\left(-\frac{cr_{ps}\cos\theta_{ps}}{2\alpha}\right)\right\}K_0\left(\frac{cr_{ps}}{2\alpha}\right) \tag{11}$$

where c is the crack velocity, K is the coefficient of the thermal conductivity of the material, α is the coefficient of thermal diffusivity, K_0 is the second-kind zero-order Bessel function and $dq(r_s, \theta_s)$ is the heat density of the point source $S(r_s, \theta_s)$ given by eqn (10). Moreover, (r_{ps}, θ_{ps}) are the relative coordinates of the point $P(r_p, \theta_p)$ referred to the elementary source $S(r_s, \theta_s)$, which are derived and given by simple geometrical relations as it is clear in Fig. 1.

Integrating eqn (11) over the whole surface of the plastic enclave around the moving crack tip we obtain the following equation:

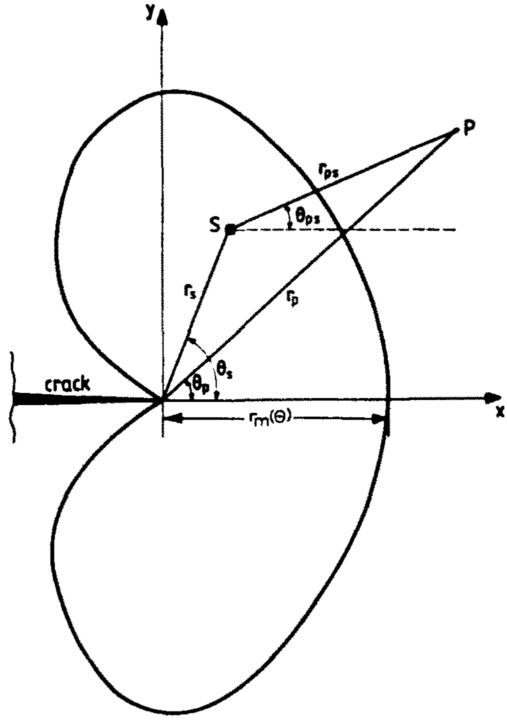

Fig. 1. The von Mises plastic enclave at the tip of a running crack as the heat source.

$$\Delta T(r_p,\theta_p) = \iint_A \frac{Qc\left[\dfrac{r_m - r_s}{r_m + r_s}\right]\cos(\theta/2)}{2\pi K(3-4\ln 2)\displaystyle\int_0^\pi r_m^2\cos(\theta/2)\,d\theta}\left\{\exp\left(\frac{cr_{ps}\cos\theta_{ps}}{2\alpha}\right)\right\}K_0\left(\frac{cr_{ps}}{2\alpha}\right)dA \tag{12}$$

which gives the total temperature increase, $\Delta T(r_p,\theta_p)$, at the point $P(r_p,\theta_p)$ under consideration.

The numerical computation of $\Delta T(t_p, \theta_p)$ proceeds by dividing the heat source by a Cartesian grid into a number of elementary orthogonal sources, whose areas tend to zero as their number increases, and tends to infinity. These elementary sources are considered as point sources obeying Rosenthal's solution (Rosenthal, 1946). The procedure proved to be stable and accurate. Moreover, it constitutes a fast method when compared with existing double integrating rules. In the present application we have used $N = 100$ as a number of partitions of the heat source, along either direction, thus dividing the heat source into about 10^4 elementary sources. The maximum error was estimated to be less than 0.5% everywhere.

3. RESULTS AND DISCUSSION

The above-described method was applied to two materials with low and high thermal conductivities, K, respectively, namely a PMMA polymer plate with $K = 0.193$ W m^{-1} °C and an aluminum alloy plate with $K = 228.45$ W m^{-1} °C. Their diffusing coefficients were $\alpha = 3.36\times10^{-7}$ m^2 s^{-1} for PMMA and $\alpha = 9.46\times10^{-5}$ m^2 s^{-1} for the aluminum alloy. Thin plates of either material were subjected to dynamic loads causing propagation of pre-existing initial cracks of a length $2\alpha = 0.15$ m under mode-I plane-stress conditions. The values of the energy, Q, dissipated as heat, were taken from Döll (1973)—see Fig. 6, for the case of PMMA—while for the case of aluminum alloy a procedure similar to that of Wells (1953) was adopted.

It was also assumed that the thin plates of both materials were loaded by a stress at infinity equal to $0.2\sigma_0$, where σ_0 is the yield stress in simple tension for the respective material. Consequently, temperature elevations computed by the present method depend on the stress level assumed. However, due to the linear elastic stress field description incorporated, isothermal contours are self-similar versus applied load.

Temperature elevations at point $P(r_p, \theta_p)$ lying in a narrow band very close to the elastic–plastic boundary $r_m(\theta)$ were computed. This band has a width corresponding to $1.01\ r_m(\theta) \leqslant r_p \leqslant 1.02\ r_m(\theta)$. Even in such a narrow band temperature gradients evaluated by the model were significant, especially for the case of PMMA.

Figure 2 presents the isothermal contours for the PMMA plate plotted around the crack tip for three different crack velocities and for a narrow zone surrounding the heat source and extending between the radii $1.01\ r_m(\theta)$ and $1.02\ r_m(\theta)$, where $r_m(\theta)$ is the respective polar distance of the elastic–plastic boundary for either material. In Fig. 2b this boundary is also plotted under a reduced scale in order to avoid confusion with the isothermals lying inside this zone. One can observe in these figures that:

(i) Strong temperature gradients exist in both radial and angular directions. Thus, for crack velocities $c = 0.50\ c_2$ and for $(r_p, \theta_p) \approx (1.02\ r_m, 0^\circ)$ there is a gradient $\Delta T/\Delta r \approx 1.3 \times 10^8\ ^\circ\mathrm{C\ m^{-1}}$, while for $(r_p, \theta_p) \approx (1.02\ r_m, 90^\circ)$ this gradient becomes equal to $\Delta T/\Delta r \approx 1.9 \times 10^8\ ^\circ\mathrm{C\ m^{-1}}$, that is, there is an increase of 40%.

(ii) For relatively low crack velocities ($c < 0.5c_2$) the temperature distribution shows a single minimum at $\theta_p = 0^\circ$ and a maximum (not shown in Fig. 2) at $\theta_p = 180^\circ$, as it was also concluded by Weichert and Schönert (1974, 1978). For velocities higher than half the shear wave velocity, $c > 0.5c_2$, two symmetric minima appear at a maximum θ_p value up to about 50°.

Figure 3 presents the isothermal contours for aluminum plates. It is readily concluded that remarks (i) and (ii) above, valid for the case of PMMA plates, are also valid for the aluminum plates, but now the gradients are drastically reduced. This is a reasonable conclusion because of the high thermal conductivity of the aluminum.

Figure 4 presents the variation of the temperature versus the polar angle θ_p for aluminum plates, plotted for a series of different crack velocities. The interesting remark derived from this figure is that, for low crack velocities, two symmetric minima exist with respect to the crack axis, while for high crack velocities and along the same directions the two minima invert into local maxima.

Concerning, now, the influence of the exponent n on the temperature elevations, it can be readily derived that as n increases, the temperature outside the plastic enclave is also increasing. This behavior is reasonable since for a given total heat production Q (eqn 7), low values of n result in strongly varying heat distributions inside the plastic enclave, while high values of n result in a more homogeneous situation.

The influence of the exponent n can be seen in Fig. 5 where for $n = 0.25$ most of the total heat is produced very close to the crack tip, whereas for $n = 4$, almost up to the half of the plastic radius ($r_s/r_m = 0.4$) heat production is roughly equal to that of the crack tip. So a generic stationary point P outside the plastic enclave in the former case is weakly affected by the rather distant highly productive elementary heat sources, whereas in the latter case with $n = 4.0$, these "very hot" sources approach closer to the point P and so their influence is stronger.

Figure 6 presents the relative variation of the temperature increase for a generic stationary point ahead of the crack tip ($\theta_p = 0^\circ$) lying at a distance $r_p = 1.01 r_m$ for PMMA and aluminum versus the n values, reduced to their values for $n = 1$. It is clear that, as n increases, temperatures also increase in accordance with the previous conclusions. In addition, the poor heat conductivity of PMMA causes much higher temperature increases in comparison to those for aluminum, presenting good heat conductivity.

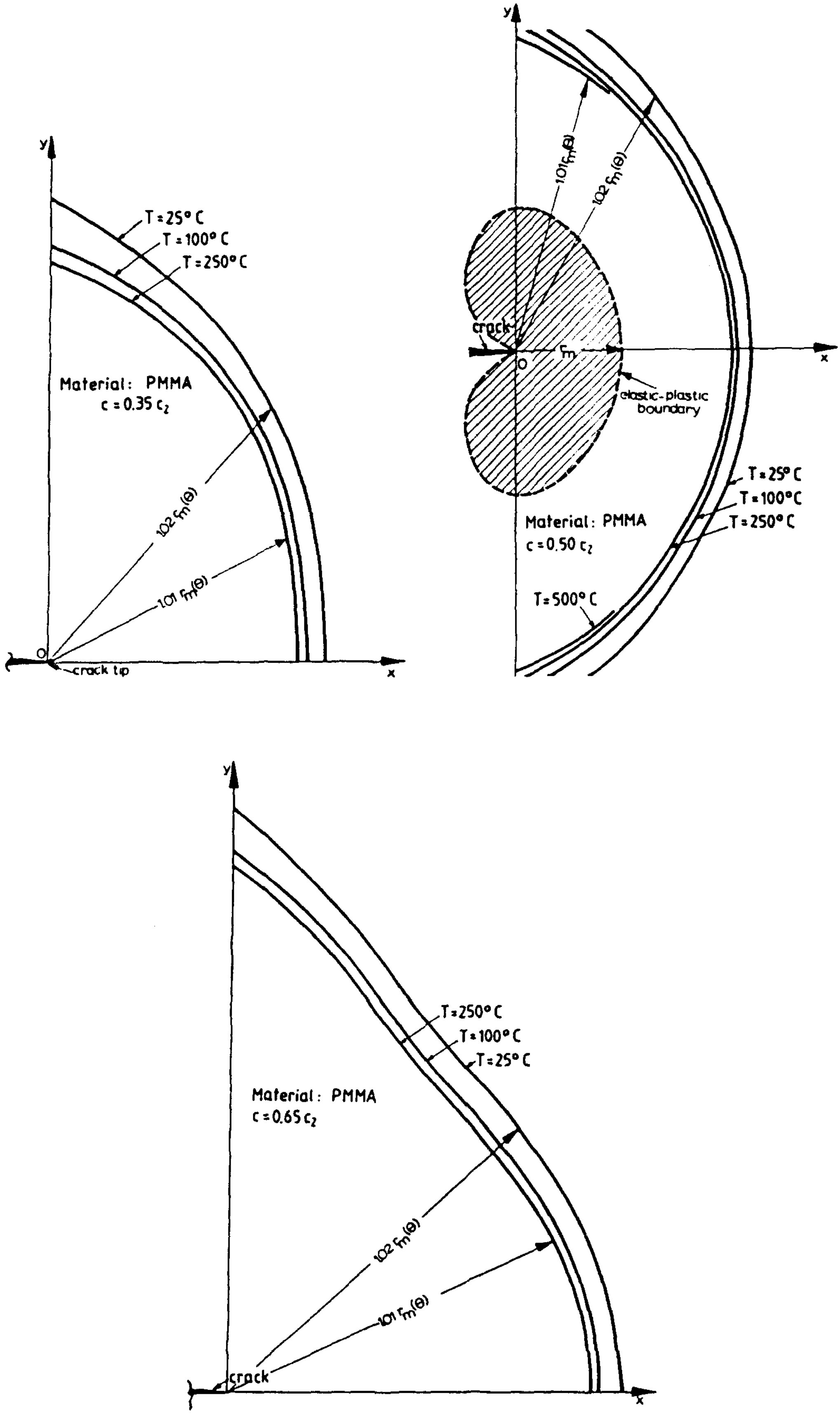

Fig. 2. Isothermal contours surrounding crack tip for the case of PMMA for three different crack velocities: $c = 0.35$, $c = 0.50$, $c = 0.65$.

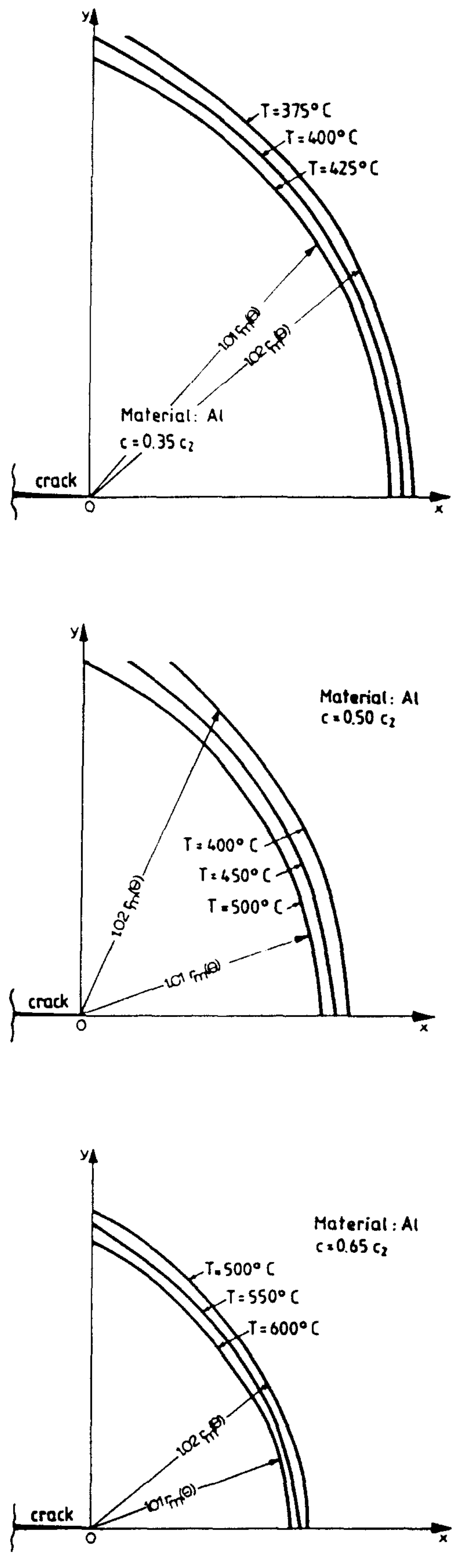

Fig. 3. Isothermal contours surrounding crack tip for the case of aluminum for three different crack velocities: $c = 0.35$, $c = 0.50$, $c = 0.65$.

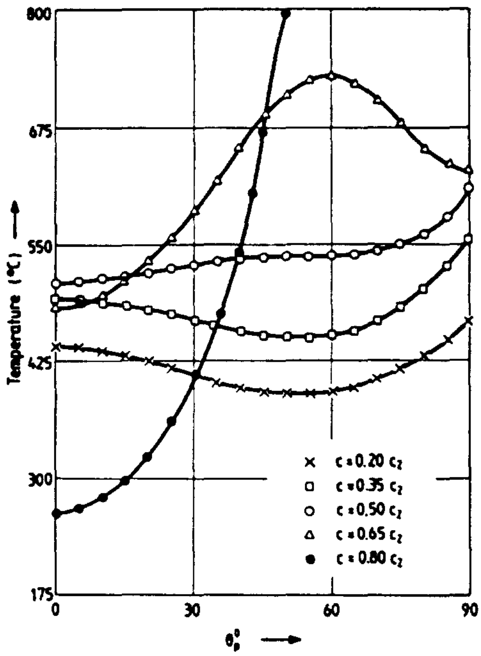

Fig. 4. The variation of temperature for the case of aluminum, at a distance $r(\theta) = 1.01\, r_m(\theta)$ from the crack tip, versus angle θ_p, for different crack velocities.

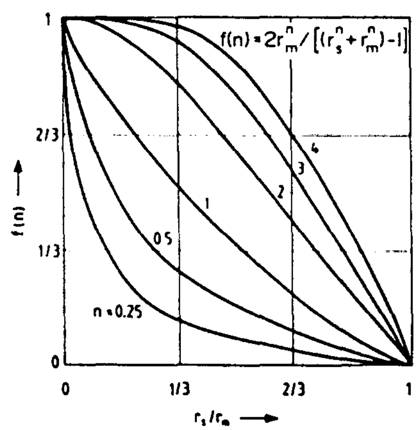

Fig. 5. Variation of the function $f(n)$ of eqn (3) versus reduced elementary heat sources r_s/r_m for various values of exponent n.

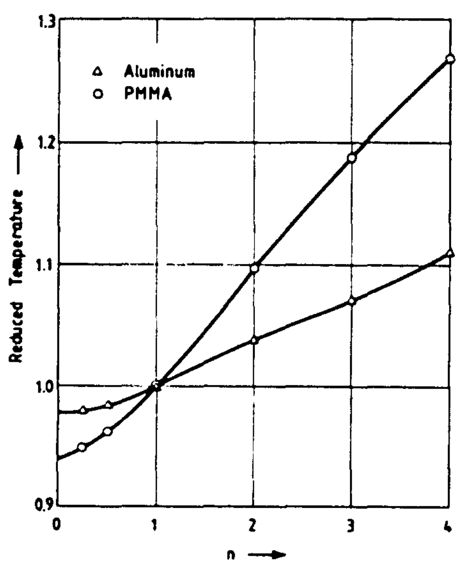

Fig. 6. Temperature variation at a generic point ahead of the crack tip ($\theta_p = 0°$) at a distance $r_p = 1.01\, r_m$, versus exponent n for two materials.

4. CONCLUSIONS

The conclusions of the present paper may be summarized as follows:

(i) The theoretically predicted temperature elevations are significant and they agree well with existing experimental evidence.

(ii) The improved hypotheses on which the present model is based, elucidated a hitherto hidden relationship between temperature distribution and the crack-branching phenomenon. It is interesting to observe the close coincidences between bifurcation angles and local temperature minima, which suggest that the propagating cracks prefer to proceed along the two paths where the material is less ductile.

The results of the present model may be further improved by taking into consideration the phenomenon of heat transfer by means of heat conduction, given that the heat being lost by radiation seems to be negligible.

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